

# NAG Library Routine Document

## G10ABF

**Note:** before using this routine, please read the Users' Note for your implementation to check the interpretation of *bold italicised* terms and other implementation-dependent details.

### 1 Purpose

G10ABF fits a cubic smoothing spline for a given smoothing parameter.

### 2 Specification

```

SUBROUTINE G10ABF (MODE, WEIGHT, N, X, Y, WT, RHO, YHAT, C, LDC, RSS,      &
                  DF, RES, H, COMM, IFAIL)
INTEGER          N, LDC, IFAIL
REAL (KIND=nag_wp) X(N), Y(N), WT(*), RHO, YHAT(N), C(LDC,3), RSS, DF,      &
                  RES(N), H(N), COMM(9*N+14)
CHARACTER(1)    MODE, WEIGHT

```

### 3 Description

G10ABF fits a cubic smoothing spline to a set of  $n$  observations  $(x_i, y_i)$ , for  $i = 1, 2, \dots, n$ . The spline provides a flexible smooth function for situations in which a simple polynomial or nonlinear regression model is unsuitable.

Cubic smoothing splines arise as the unique real-valued solution function  $f$ , with absolutely continuous first derivative and squared-integrable second derivative, which minimizes:

$$\sum_{i=1}^n w_i (y_i - f(x_i))^2 + \rho \int_{-\infty}^{\infty} (f''(x))^2 dx,$$

where  $w_i$  is the (optional) weight for the  $i$ th observation and  $\rho$  is the smoothing parameter. This criterion consists of two parts: the first measures the fit of the curve, and the second the smoothness of the curve. The value of the smoothing parameter  $\rho$  weights these two aspects; larger values of  $\rho$  give a smoother fitted curve but, in general, a poorer fit. For details of how the cubic spline can be estimated see Hutchinson and de Hoog (1985) and Reinsch (1967).

The fitted values,  $\hat{y} = (\hat{y}_1, \hat{y}_2, \dots, \hat{y}_n)^T$ , and weighted residuals,  $r_i$ , can be written as

$$\hat{y} = Hy \quad \text{and} \quad r_i = \sqrt{w_i}(y_i - \hat{y}_i)$$

for a matrix  $H$ . The residual degrees of freedom for the spline is  $\text{trace}(I - H)$  and the diagonal elements of  $H$ ,  $h_{ii}$ , are the leverages.

The parameter  $\rho$  can be chosen in a number of ways. The fit can be inspected for a number of different values of  $\rho$ . Alternatively the degrees of freedom for the spline, which determines the value of  $\rho$ , can be specified, or the (generalized) cross-validation can be minimized to give  $\rho$ ; see G10ACF for further details.

G10ABF requires the  $x_i$  to be strictly increasing. If two or more observations have the same  $x_i$ -value then they should be replaced by a single observation with  $y_i$  equal to the (weighted) mean of the  $y$  values and weight,  $w_i$ , equal to the sum of the weights. This operation can be performed by G10ZAF.

The computation is split into three phases.

- (i) Compute matrices needed to fit spline.
- (ii) Fit spline for a given value of  $\rho$ .
- (iii) Compute spline coefficients.

When fitting the spline for several different values of  $\rho$ , phase (i) need only be carried out once and then phase (ii) repeated for different values of  $\rho$ . If the spline is being fitted as part of an iterative weighted least squares procedure phases (i) and (ii) have to be repeated for each set of weights. In either case, phase (iii) will often only have to be performed after the final fit has been computed.

The algorithm is based on Hutchinson (1986).

## 4 References

Hastie T J and Tibshirani R J (1990) *Generalized Additive Models* Chapman and Hall

Hutchinson M F (1986) Algorithm 642: A fast procedure for calculating minimum cross-validation cubic smoothing splines *ACM Trans. Math. Software* **12** 150–153

Hutchinson M F and de Hoog F R (1985) Smoothing noisy data with spline functions *Numer. Math.* **47** 99–106

Reinsch C H (1967) Smoothing by spline functions *Numer. Math.* **10** 177–183

## 5 Arguments

- 1: MODE – CHARACTER(1) *Input*  
*On entry:* indicates in which mode the routine is to be used.  
 MODE = 'P'  
 Initialization and fitting is performed. This partial fit can be used in an iterative weighted least squares context where the weights are changing at each call to G10ABF or when the coefficients are not required.  
 MODE = 'Q'  
 Fitting only is performed. Initialization must have been performed previously by a call to G10ABF with MODE = 'P'. This quick fit may be called repeatedly with different values of RHO without re-initialization.  
 MODE = 'F'  
 Initialization and full fitting is performed and the function coefficients are calculated.  
*Constraint:* MODE = 'P', 'Q' or 'F'.
- 2: WEIGHT – CHARACTER(1) *Input*  
*On entry:* indicates whether user-defined weights are to be used.  
 WEIGHT = 'W'  
 User-defined weights should be supplied in WT.  
 WEIGHT = 'U'  
 The data is treated as unweighted.  
*Constraint:* WEIGHT = 'W' or 'U'.
- 3: N – INTEGER *Input*  
*On entry:*  $n$ , the number of distinct observations.  
*Constraint:*  $N \geq 3$ .
- 4: X(N) – REAL (KIND=nag\_wp) array *Input*  
*On entry:* the distinct and ordered values  $x_i$ , for  $i = 1, 2, \dots, n$ .  
*Constraint:*  $X(i) < X(i + 1)$ , for  $i = 1, 2, \dots, n - 1$ .
- 5: Y(N) – REAL (KIND=nag\_wp) array *Input*  
*On entry:* the values  $y_i$ , for  $i = 1, 2, \dots, n$ .

- 6: WT(\*) – REAL (KIND=nag\_wp) array Input  
**Note:** the dimension of the array WT must be at least N if WEIGHT = 'W'.  
*On entry:* if WEIGHT = 'W', WT must contain the  $n$  weights. Otherwise WT is not referenced and unit weights are assumed.  
*Constraint:* if WEIGHT = 'W',  $WT(i) > 0.0$ , for  $i = 1, 2, \dots, n$ .
- 7: RHO – REAL (KIND=nag\_wp) Input  
*On entry:*  $\rho$ , the smoothing parameter.  
*Constraint:*  $RHO \geq 0.0$ .
- 8: YHAT(N) – REAL (KIND=nag\_wp) array Output  
*On exit:* the fitted values,  $\hat{y}_i$ , for  $i = 1, 2, \dots, n$ .
- 9: C(LDC, 3) – REAL (KIND=nag\_wp) array Input/Output  
*On entry:* if MODE = 'Q', C must be unaltered from the previous call to G10ABF with MODE = 'P'. Otherwise C need not be set.  
*On exit:* if MODE = 'F', C contains the spline coefficients. More precisely, the value of the spline at  $t$  is given by  $((C(i, 3) \times d + C(i, 2)) \times d + C(i, 1)) \times d + \hat{y}_i$ , where  $x_i \leq t < x_{i+1}$  and  $d = t - x_i$ .  
 If MODE = 'P' or 'Q', C contains information that will be used in a subsequent call to G10ABF with MODE = 'Q'.
- 10: LDC – INTEGER Input  
*On entry:* the first dimension of the array C as declared in the (sub)program from which G10ABF is called.  
*Constraint:*  $LDC \geq N - 1$ .
- 11: RSS – REAL (KIND=nag\_wp) Output  
*On exit:* the (weighted) residual sum of squares.
- 12: DF – REAL (KIND=nag\_wp) Output  
*On exit:* the residual degrees of freedom.
- 13: RES(N) – REAL (KIND=nag\_wp) array Output  
*On exit:* the (weighted) residuals,  $r_i$ , for  $i = 1, 2, \dots, n$ .
- 14: H(N) – REAL (KIND=nag\_wp) array Output  
*On exit:* the leverages,  $h_{ii}$ , for  $i = 1, 2, \dots, n$ .
- 15: COMM(9 × N + 14) – REAL (KIND=nag\_wp) array Communication Array  
*On entry:* if MODE = 'Q', COMM must be unaltered from the previous call to G10ABF with MODE = 'P'. Otherwise COMM need not be set.  
*On exit:* if MODE = 'P' or 'Q', COMM contains information that will be used in a subsequent call to G10ABF with MODE = 'Q'.
- 16: IFAIL – INTEGER Input/Output  
*On entry:* IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this argument you should refer to Section 3.4 in How to Use the NAG Library and its Documentation for details.

For environments where it might be inappropriate to halt program execution when an error is detected, the value  $-1$  or  $1$  is recommended. If the output of error messages is undesirable, then the value  $1$  is recommended. Otherwise, if you are not familiar with this argument, the recommended value is  $0$ . **When the value  $-1$  or  $1$  is used it is essential to test the value of IFAIL on exit.**

*On exit:* IFAIL =  $0$  unless the routine detects an error or a warning has been flagged (see Section 6).

## 6 Error Indicators and Warnings

If on entry IFAIL =  $0$  or  $-1$ , explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL =  $1$

On entry,  $N < 3$ ,  
 or  $LDC < N - 1$ ,  
 or  $RHO < 0.0$ ,  
 or  $MODE \neq 'Q', 'P' \text{ or } 'F'$ ,  
 or  $WEIGHT \neq 'W' \text{ or } 'U'$ .

IFAIL =  $2$

On entry,  $WEIGHT = 'W'$  and at least one element of  $WT \leq 0.0$ .

IFAIL =  $3$

On entry,  $X(i) \geq X(i + 1)$ , for some  $i$ ,  $i = 1, 2, \dots, n - 1$ .

IFAIL =  $-99$

An unexpected error has been triggered by this routine. Please contact NAG.

See Section 3.9 in How to Use the NAG Library and its Documentation for further information.

IFAIL =  $-399$

Your licence key may have expired or may not have been installed correctly.

See Section 3.8 in How to Use the NAG Library and its Documentation for further information.

IFAIL =  $-999$

Dynamic memory allocation failed.

See Section 3.7 in How to Use the NAG Library and its Documentation for further information.

## 7 Accuracy

Accuracy depends on the value of  $\rho$  and the position of the  $x$  values. The values of  $x_i - x_{i-1}$  and  $w_i$  are scaled and  $\rho$  is transformed to avoid underflow and overflow problems.

## 8 Parallelism and Performance

G10ABF is not threaded in any implementation.

## 9 Further Comments

The time taken by G10ABF is of order  $n$ .

Regression splines with a small ( $< n$ ) number of knots can be fitted by E02BAF and E02BEF.

## 10 Example

The data, given by Hastie and Tibshirani (1990), is the age,  $x_i$ , and C-peptide concentration (pmol/ml),  $y_i$ , from a study of the factors affecting insulin-dependent diabetes mellitus in children. The data is input, reduced to a strictly ordered set by G10ZAF and a series of splines fit using a range of values for the smoothing parameter,  $\rho$ .

### 10.1 Program Text

```

Program g10abfe

!      G10ABF Example Program Text

!      Mark 26 Release. NAG Copyright 2016.

!      .. Use Statements ..
Use nag_library, Only: g10abf, g10zaf, nag_wp
!      .. Implicit None Statement ..
Implicit None
!      .. Parameters ..
Integer, Parameter          :: nin = 5, nout = 6
!      .. Local Scalars ..
Integer                     :: i, ifail, j, lcomm, ldc, lwt, n,      &
                             nord, nrho
Character (1)               :: mode, weight
!      .. Local Arrays ..
Real (Kind=nag_wp), Allocatable :: c(:, :), comm(:), df(:), h(:),      &
                             res(:), rho(:), rss(:), wt(:),          &
                             wwt(:), x(:), xord(:), y(:),          &
                             yhat(:, :), yord(:)
Integer, Allocatable        :: iwrk(:)
!      .. Executable Statements ..
Write (nout,*) ' G10ABF Example Program Results'
Write (nout,*)

!      Skip heading in data file
Read (nin,*)

!      Read in problem size and control parameters
Read (nin,*) n, weight, nrho

If (weight=='W' .Or. weight=='w') Then
  lwt = n
Else
  lwt = 0
End If
lcomm = 9*n + 14
ldc = n - 1
Allocate (x(n),y(n),wt(lwt),xord(n),yord(n),wwt(n),yhat(n,nrho),      &
         c(ldc,3),res(n),h(n),comm(lcomm),iwrk(n),rho(nrho),rss(nrho),df(nrho))

!      Read in the smoothing parameters
Read (nin,*) rho(1:nrho)

!      Read in data
If (lwt>0) Then
  Read (nin,*)(x(i),y(i),wt(i),i=1,n)
Else
  Read (nin,*)(x(i),y(i),i=1,n)
End If

!      Reorder data into increasing X and remove tied observations, weighting
!      accordingly
ifail = 0
Call g10zaf(weight,n,x,y,wt,nord,xord,yord,wwt,rss(1),iwrk,ifail)

!      Fit cubic spline the first time
!      NB: These are weighted as G10ZAF creates weights
ifail = 0

```

```

mode = 'P'
Call g10abf(mode,'W',nord,xord,yord,wwt,rho(1),yhat(1,1),c,ldc,rss(1), &
  df(1),res,h,comm,ifail)

!   Fit cubic spline the remaining NRHO - 1 times
mode = 'Q'
Do i = 2, nrho
  ifail = 0
  Call g10abf(mode,'W',nord,xord,yord,wwt,rho(i),yhat(1,i),c,ldc,rss(i), &
    df(i),res,h,comm,ifail)
End Do

!   Display results
Write (nout,99999) 'Smoothing coefficient (rho) = ', rho(1:nrho)
Write (nout,99998) 'Residual sum of squares      = ', rss(1:nrho)
Write (nout,99998) 'Degrees of freedom          = ', df(1:nrho)
Write (nout,*)
Write (nout,*) '      X      Y      Fitted Values'
Do i = 1, nord
  Write (nout,99997) xord(i), yord(i), (yhat(i,j),j=1,nrho)
End Do

99999 Format (1X,A,10(2X,F8.2))
99998 Format (1X,A,10(F10.3))
99997 Format (1X,2F8.4,14X,10(2X,F8.4))
End Program g10abfe

```

## 10.2 Program Data

```

G10ABF Example Program Data
43 'U' 3      :: N,MODE,WEIGHT,NRHO
1.0 10.0 100.0 :: RHO
  5.2  4.8
  8.8  4.1
10.5  5.2
10.6  5.5
10.4  5.0
  1.8  3.4
12.7  3.4
15.6  4.9
  5.8  5.6
  1.9  3.7
  2.2  3.9
  4.8  4.5
  7.9  4.8
  5.2  4.9
  0.9  3.0
11.8  4.6
  7.9  4.8
11.5  5.5
10.6  4.5
  8.5  5.3
11.1  4.7
12.8  6.6
11.3  5.1
  1.0  3.9
14.5  5.7
11.9  5.1
  8.1  5.2
13.8  3.7
15.5  4.9
  9.8  4.8
11.0  4.4
12.4  5.2
11.1  5.1
  5.1  4.6
  4.8  3.9
  4.2  5.1
  6.9  5.1
13.2  6.0

```

```

9.9  4.9
12.5 4.1
13.2 4.6
 8.9 4.9
10.8 5.1      :: End of X,Y

```

### 10.3 Program Results

G10ABF Example Program Results

```

Smoothing coefficient (rho) =      1.00      10.00      100.00
Residual sum of squares    =      9.118     11.288     11.881
Degrees of freedom         =     22.505     27.785     31.191

```

X	Y	Fitted Values		
0.9000	3.0000	3.3784	3.3674	3.3699
1.0000	3.9000	3.4173	3.4008	3.4063
1.8000	3.4000	3.6144	3.6642	3.6973
1.9000	3.7000	3.6639	3.7016	3.7341
2.2000	3.9000	3.8607	3.8214	3.8449
4.2000	5.1000	4.7441	4.5265	4.5194
4.8000	4.2000	4.4914	4.6471	4.6746
5.1000	4.6000	4.6708	4.7561	4.7470
5.2000	4.8500	4.7704	4.7993	4.7702
5.8000	5.6000	5.3426	5.0458	4.8879
6.9000	5.1000	5.1728	5.1204	4.9753
7.9000	4.8000	4.9467	4.9590	4.9537
8.1000	5.2000	4.9556	4.9262	4.9452
8.5000	5.3000	4.8742	4.8595	4.9276
8.8000	4.1000	4.7305	4.8172	4.9168
8.9000	4.9000	4.7024	4.8095	4.9143
9.8000	4.8000	4.8394	4.8676	4.9170
9.9000	4.9000	4.8746	4.8818	4.9191
10.4000	5.0000	4.9971	4.9445	4.9303
10.5000	5.2000	4.9997	4.9521	4.9321
10.6000	5.0000	4.9921	4.9572	4.9335
10.8000	5.1000	4.9603	4.9613	4.9354
11.0000	4.4000	4.9396	4.9614	4.9363
11.1000	4.9000	4.9494	4.9618	4.9366
11.3000	5.1000	4.9926	4.9623	4.9366
11.5000	5.5000	5.0116	4.9568	4.9355
11.8000	4.6000	4.9372	4.9338	4.9315
11.9000	5.1000	4.9042	4.9251	4.9300
12.4000	5.2000	4.7929	4.8943	4.9240
12.5000	4.1000	4.8042	4.8944	4.9237
12.7000	3.4000	4.9020	4.9051	4.9244
12.8000	6.6000	4.9752	4.9138	4.9252
13.2000	5.3000	5.0173	4.9239	4.9276
13.8000	3.7000	4.6164	4.8930	4.9304
14.5000	5.7000	5.1883	4.9938	4.9518
15.5000	4.9000	4.9854	4.9773	4.9687
15.6000	4.9000	4.9167	4.9682	4.9697

**Example Program**  
Cubic Smoothing Spline  
Study of the factors affecting insulin-dependent diabetes mellitus in children  
Hastie and Tibshirani (1990)

